Learning Objectives

At the end of the class you should be able to:

- devise an useful heuristic function for a problem
- demonstrate how best-first and A^* search will work on a graph
- predict the space and time requirements for best-first and A^* search

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- $h(n)$ is an underestimate if there is no path from n to a goal with cost less than $h(n)$.
- An admissible heuristic is a heuristic function that is an underestimate of the actual cost of a path to a goal.

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- If the goal is to collect all of the coins and not run out of fuel, the cost is an estimate of how many steps it will take to collect the rest of the coins, refuel when necessary, and return to goal position.
- A heuristic function can be found by solving a simpler (less constrained) version of the problem.

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- It inherits all of the advantages/disadvantages of depth-first search, but locally heads towards a goal.

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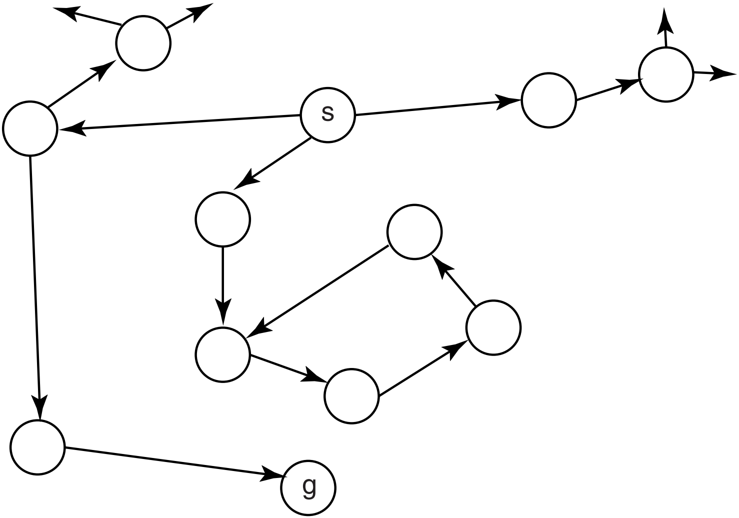
- Best-first search selects a path on the frontier with minimal h-value.
- It treats the frontier as a priority queue ordered by h.

Illustrative Graph — Heuristic Search



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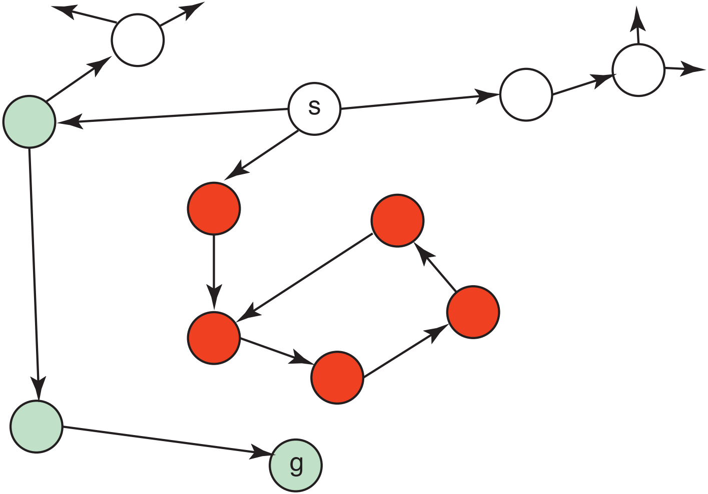

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- Let $f(p) = cost(p) + h(p)$.
  $f(p)$ estimates the total path cost of going from a start node to a goal via p.

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- In A^* search, the frontier is a priority queue ordered by $f(p)$.
- It always selects the path on the frontier with the lowest estimated cost from the start to a goal node constrained to go via that path.

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- the branching factor is finite
- arc costs are bounded above zero (there is some $\epsilon > 0$ such that all of the arc costs are greater than ϵ), and
- $h(n)$ is nonnegative and an underestimate of the cost of the shortest path from n to a goal node:

$$0 \leq h(n) \leq \text{cost of shortest path from } n \text{ to a goal}$$

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- So $cost(p) \leq cost(p'')$ for any other path p'' to a goal.

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A^* can always find a solution if there is one:

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- A^* halts, as the costs of the paths on the frontier keeps increasing, and will eventually exceed any finite number.

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How can a better heuristic function help?

Summary of Search Strategies

Strategy	Frontier Selection	Complete	Halts	Space
Depth-first	Last node added			
Breadth-first	First node added			
Heuristic depth-first	Local min $h(p)$			
Best-first	Global min $h(p)$			
Lowest-cost-first	Minimal $cost(p)$			
A^*	Minimal $f(p)$			

Complete — if there a path to a goal, it can find one, even on infinite graphs.

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Heuristic depth-first	Local min $h(p)$	No	No	Linear
Best-first	Global min $h(p)$	No	No	Exp
Lowest-cost-first	Minimal $cost(p)$	Yes	No	Exp
A^*	Minimal $f(p)$	Yes	No	Exp

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