"The mind is a neural computer, fitted by natural selection with combinatorial algorithms for causal and probabilistic reasoning about plants, animals, objects, and people."

...

"In a universe with any regularities at all, decisions informed about the past are better than decisions made at random. That has always been true, and we would expect organisms, especially informavores such as humans, to have evolved acute intuitions about probability. The founders of probability, like the founders of logic, assumed they were just formalizing common sense."

Steven Pinker, How the Mind Works, 1997, pp. 524, 343.

Learning Objectives

At the end of the class you should be able to:

- justify the use and semantics of probability
- know how to compute marginals and apply Bayes' theorem
- identify conditional independence
- build a belief network for a domain
- predict the inferences for a belief network
- explain the predictions of a causal model

Using Uncertain Knowledge

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  Example: wearing a seat belt.
- An agent needs to reason about its uncertainty.
- When an agent makes an action under uncertainty, it is gambling $\Longrightarrow$ probability.

Probability

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- An agent's belief depends on its prior belief and what it observes.
- Example: An agent's probability of a particular bird flying
  - Other agents may have different probabilities
  - An agent's belief in a bird's flying ability is affected by what the agent knows about that bird.

Random Variables

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- Assignment $X = x$ means variable X has value x.
- A proposition is a Boolean formula made from assignments of values to variables or inequality (e.g., $<$, $\leq$,...) between variables and values.

Possible World Semantics

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- $\omega \models X = x$
  means variable X is assigned value x in world ω.
- Logical connectives have their standard meaning:

$$\omega \models \alpha \wedge \beta \text{ if } \omega \models \alpha \text{ and } \omega \models \beta$$
$$\omega \models \alpha \vee \beta \text{ if } \omega \models \alpha \text{ or } \omega \models \beta$$
$$\omega \models \neg\alpha \text{ if } \omega \not\models \alpha$$

- Let Ω be the set of all possible worlds.

Semantics of Probability

Probability defines a measure on sets of possible worlds.
A probability measure is a function μ from sets of worlds into the non-negative real numbers such that:

- $\mu(\Omega) = 1$ Ω all worlds
- $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$
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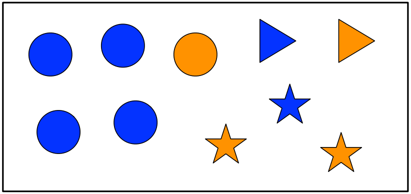
Then $P(\alpha) = \mu(\{\omega \mid \omega \models \alpha\})$.

Semantics

Possible Worlds:



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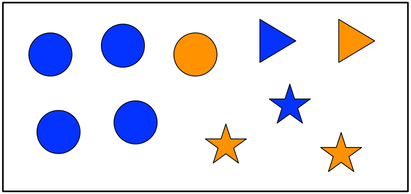

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- What is the probability of circle? $5/_{10} = 0.5$

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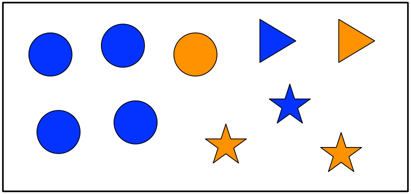

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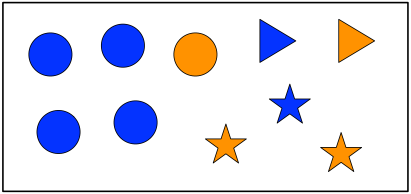

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- What is the probability of circle?
- What us the probability of star?
- What is the probability of orange?

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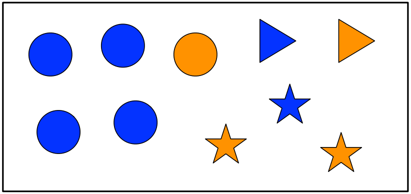

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- What us the probability of star?
- What is the probability of orange?
- What is the probability of orange and star? 2/10

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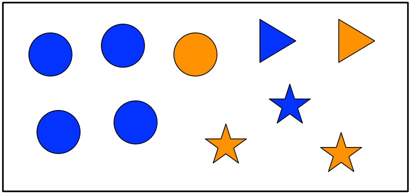

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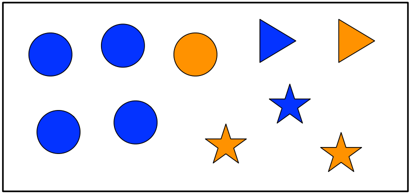

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- What is the probability of circle?
- What us the probability of star?
- What is the probability of orange?
- What is the probability of orange and star?
- What is the probability of orange and circle?
- Note that $P(\alpha \wedge \beta)$ is not a function of $P(\alpha)$ and $P(\beta)$.

Axioms of Probability (finite case)

Three axioms define what follows from a set of probabilities:

Axiom 1 $0 \leq P(a)$ for any proposition a.

Axiom 2 $P(true) = 1$

Axiom 3 $P(a \vee b) = P(a) + P(b)$ if a and b cannot both be true.

- These axioms are sound and complete with respect to the semantics.

Conditioning

- Probabilistic conditioning specifies how to revise beliefs based on new information.

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- An agent builds a probabilistic model taking all background information into account.
  This gives the prior probability.
- All other information must be conditioned on.
- If evidence e is the all of the information obtained subsequently, the conditional probability $P(h \mid e)$ of h given e is the posterior probability of h.

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$$\mu_e(S) = \begin{cases} & \text{if } \omega \not\models e \text{ for all } \omega \in S \end{cases}$$

e is false in all worlds in S.

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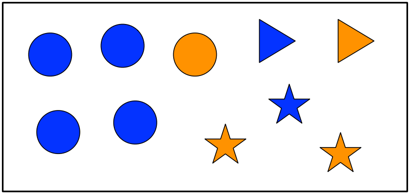

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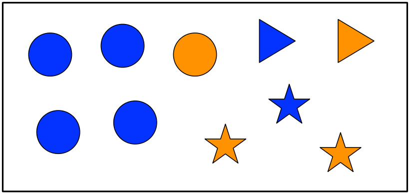

Observe *Color*=*orange*:

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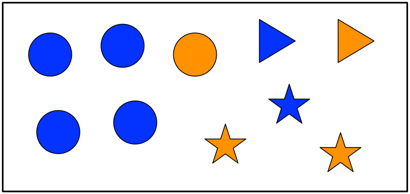

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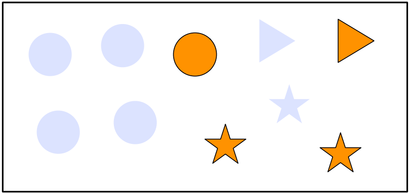

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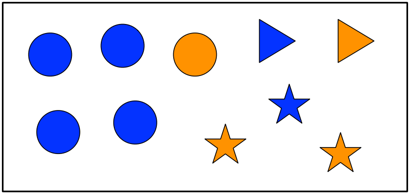

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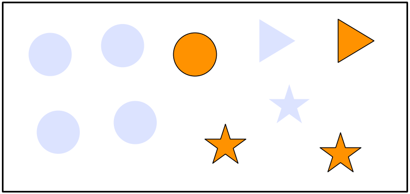

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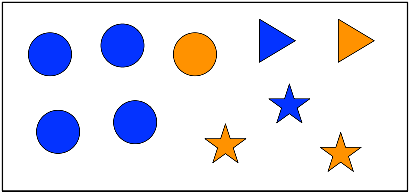

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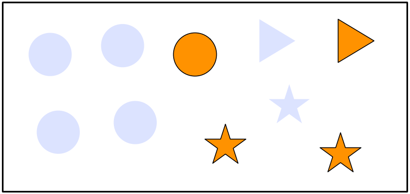

$$P(Shape{=}star \mid Color{=}orange) = 0.5$$

$$P(Shape{=}circle \mid Color{=}orange) = 0.25$$

Exercise

Flu	*Sneeze*	*Snore*	μ
true	true	true	0.064
true	true	false	0.096
true	false	true	0.016
true	false	false	0.024
false	true	true	0.096
false	true	false	0.144
false	false	true	0.224
false	false	false	0.336

What is:

(a) $P(flu \wedge sneeze)$

(b) $P(flu \wedge \neg sneeze)$

(c) $P(flu)$ top c

(d) $P(sneeze \mid flu)$ a/c

(e) $P(\neg flu \wedge sneeze)$

(f) $P(flu \mid sneeze)$

(g) $P(sneeze \mid flu \wedge snore)$

(h) $P(flu \mid sneeze \wedge snore)$

Chain Rule

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$$P(\overset{h}{f_n} \land \overbrace{f_{n-1} \land \ldots \land f_1}^{e})$$

$$=$$

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$$
\begin{aligned}
& P(f_n \wedge f_{n-1} \wedge \ldots \wedge f_1) \\
& \quad = \quad P(f_n \mid f_{n-1} \wedge \cdots \wedge f_1) \times \\
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&\qquad \times \cdots \times P(f_3 \mid f_2 \wedge f_1) \times P(f_2 \mid f_1) \times P(f_1) \\
&\quad = \prod_{i=1}^{n} P(f_i \mid f_1 \wedge \cdots \wedge f_{i-1})
\end{aligned}
$$

Bayes' theorem

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$$P(h \mid e) = \frac{P(e \mid h) \times P(h)}{P(e)}.$$

This is Bayes' theorem.

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  $P(\mathit{fire} \mid \mathit{alarm})$
  $P($*a tree is in front of a car* $\mid$ *image looks like* $)$

Exercise

A cab was involved in a hit-and-run accident at night. Two cab companies, the Green and the Blue, operate in the city. You are given the following data:

- 85% of the cabs in the city are Green and 15% are Blue.
- A witness identified the cab as Blue. The court tested the reliability of the witness in the circumstances that existed on the night of the accident and concluded that the witness correctly identifies each one of the two colours 80% of the time and failed 20% of the time.

What is the probability that the cab involved in the accident was Blue?

[From D. Kahneman, Thinking Fast and Slow, 2011, p. 166.]