Simple language: propositional definite clauses

Propositional definite clauses are a resticited form of propostions that can't represent disjunction of atoms:

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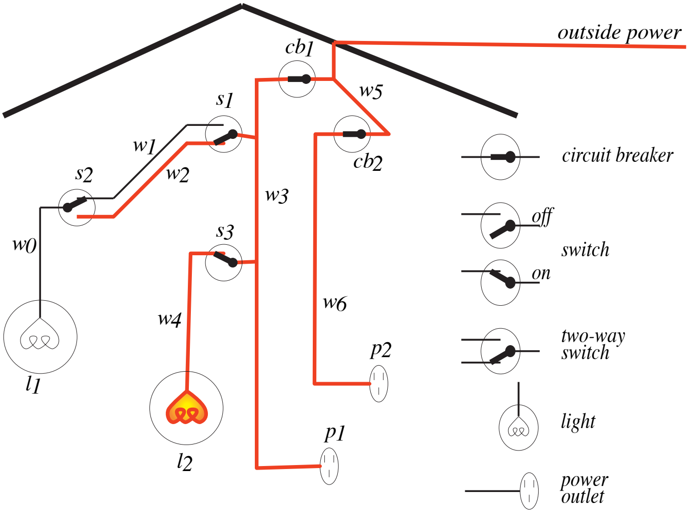
- A qeury is a body that is asked of a knowledge base.

Electrical Environment



Representing the Electrical Environment

$light\_l_1.$

$light\_l_2.$

$down\_s_1.$

$up\_s_2.$

$up\_s_3.$

$ok\_l_1.$

$ok\_l_2.$

$ok\_cb_1.$

$ok\_cb_2.$

$live\_outside.$

$lit\_l_1 \leftarrow live\_w_0 \land ok\_l_1$

$live\_w_0 \leftarrow live\_w_1 \land up\_s_2.$

$live\_w_0 \leftarrow live\_w_2 \land down\_s_2.$

$live\_w_1 \leftarrow live\_w_3 \land up\_s_1.$

$live\_w_2 \leftarrow live\_w_3 \land down\_s_1.$

$lit\_l_2 \leftarrow live\_w_4 \land ok\_l_2.$

$live\_w_4 \leftarrow live\_w_3 \land up\_s_3.$

$live\_p_1 \leftarrow live\_w_3.$

$live\_w_3 \leftarrow live\_w_5 \land ok\_cb_1.$

$live\_p_2 \leftarrow live\_w_6.$

$live\_w_6 \leftarrow live\_w_5 \land ok\_cb_2.$

$live\_w_5 \leftarrow live\_outside.$

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- A proof procedure is complete if $KB \models g$ implies $KB \vdash g$.
  - A complete proof procedure can produce all results.

Aside: Gödel's incompleteness theorem

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- If it is false then system is unsound.

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- If it is false then system is unsound.
- The alternative is that statement cannot be represented.
- the state of a computer can be seen as a (big) integer, and all operations as arithmetic operations
- We can write a proof system that can represent that statement in a computer.

Bottom-up Proof Procedure

One rule of derivation, a generalized form of *modus ponens*:

> *If "$h \leftarrow b_1 \wedge \ldots \wedge b_m$" is a clause in the knowledge base, and each b_i has been derived, then h can be derived.*

This is forward chaining on this clause.
(An atomic fact is treated as a clause with empty body ($m = 0$).)

Bottom-up proof procedure

$KB \vdash g$ if $g \in C$ at the end of this procedure:

$C := \{\}$;
repeat
    select fact h or rule "$h \leftarrow b_1 \wedge \ldots \wedge b_m$" in KB such that
        $b_i \in C$ for all i, and
        $h \notin C$;
    $C := C \cup \{h\}$
until no more clauses can be selected.

Example

$$a \leftarrow b \wedge c.$$
$$a \leftarrow e \wedge f.$$
$$b \leftarrow f \wedge k.$$
$$c \leftarrow e.$$
$$d \leftarrow k.$$
$$e.$$
$$f \leftarrow j \wedge e.$$
$$f \leftarrow c.$$
$$j \leftarrow c.$$

Clicker Question

Consider the knowledge base KB:

$happy \leftarrow good.$ $\quad$ $foo \leftarrow bar \wedge fun.$

$happy \leftarrow green.$ $\quad$ $bar \leftarrow zed.$

$green.$ $\quad$ $zed.$

What is the final consequence set in the bottom-up proof procedure run on KB?

A $\{happy, good, green, foo, bar, fun, zed\}$

B $\{happy, good, green, foo, bar, zed\}$

C $\{happy, green, bar, zed\}$

D $\{green, bar, zed\}$

E None of the above

Soundness of bottom-up proof procedure

If $KB \vdash g$ then $KB \models g$.

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  where each b_i is in C, and so true in I.
  h is false in I (by assumption)
  So this clause is false in I.
  Therefore I isn't a model of KB.
- Contradiction. Therefore there cannot be such a g.

Fixed Point

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- I is called a Minimal Model.

Completeness

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If $KB \models g$ then $KB \vdash g$.

- Suppose $KB \models g$. Then g is true in all models of KB.
- Thus g is true in the minimal model.
- Thus g is in the fixed point.
- Thus g is generated by the bottom up algorithm.
- Thus $KB \vdash g$.

Clicker Question

Suppose there at some atom *aaa* such that
$KB \vdash aaa$ and
$KB \not\models aaa$.
What can be inferred?

- A The proof procedure is not sound
- B The proof prodecure is not complete
- C The proof procedure is sound and complete
- D The proof procedure is either sound or complete
- E None of the above

Top-down Definite Clause Proof Procedure

Idea: search backward from a query to determine if it is a logical consequence of KB.
An answer clause is of the form:

$$yes \leftarrow a_1 \wedge a_2 \wedge \ldots \wedge a_m$$

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The SLD Resolution of this answer clause on atom a_i with the clause:

$$a_i \leftarrow b_1 \wedge \ldots \wedge b_p$$

is the answer clause

$$yes \leftarrow a_1 \wedge \ldots \wedge a_{i-1} \wedge b_1 \wedge \cdots \wedge b_p \wedge a_{i+1} \wedge \cdots \wedge a_m.$$

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An atomic fact in the knowledge base is considered as a clause where $p = 0$.

Derivations

- An answer is an answer clause with $m = 0$.
  That is, it is the answer clause $yes \leftarrow$.
- A derivation of query "$?q_1 \wedge \ldots \wedge q_k$" from KB is a sequence of answer clauses $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that
  - γ_0 is the answer clause $yes \leftarrow q_1 \wedge \ldots \wedge q_k$
  - γ_i is obtained by resolving γ_{i-1} with a clause in KB
  - γ_n is an answer.

Top-down definite clause interpreter

To solve the query $?q_1 \wedge \ldots \wedge q_k$:

> $ac :=$ "$yes \leftarrow q_1 \wedge \ldots \wedge q_k$"
> **repeat**
>   **select** atom a_i from the body of ac
>   **choose** clause C from KB with a_i as head
>   replace a_i in the body of ac by the body of C
> **until** ac is an answer.

Nondeterministic Choice

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  "select"

Nondeterministic Choice

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  "select"
- Don't-know nondeterminism If one choice doesn't lead to a solution, other choices may.
  choose

Example: successful derivation

$$
\begin{array}{lll}
a \leftarrow b \wedge c. & a \leftarrow e \wedge f. & b \leftarrow f \wedge k. \\
c \leftarrow e. & d \leftarrow k. & e. \\
f \leftarrow j \wedge e. & f \leftarrow c. & j \leftarrow c.
\end{array}
$$

Query: $?a$

Example: successful derivation

$$a \leftarrow b \wedge c. \qquad a \leftarrow e \wedge f. \qquad b \leftarrow f \wedge k.$$
$$c \leftarrow e. \qquad d \leftarrow k. \qquad e.$$
$$f \leftarrow j \wedge e. \qquad f \leftarrow c. \qquad j \leftarrow c.$$

Query: $?a$

$$\gamma_0 : \quad yes \leftarrow a \qquad \gamma_4 : \quad yes \leftarrow e$$
$$\gamma_1 : \quad yes \leftarrow e \wedge f \qquad \gamma_5 : \quad yes \leftarrow$$
$$\gamma_2 : \quad yes \leftarrow f$$
$$\gamma_3 : \quad yes \leftarrow c$$

Example: failing derivation

$$a \leftarrow b \wedge c. \qquad a \leftarrow e \wedge f. \qquad b \leftarrow f \wedge k.$$
$$c \leftarrow e. \qquad d \leftarrow k. \qquad e.$$
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Search Graph for SLD Resolution

$a \leftarrow b \wedge c.$
$a \leftarrow h.$
$b \leftarrow k.$
$d \leftarrow p.$
$sf \leftarrow p.$
$g \leftarrow f.$
$h \leftarrow m.$
$?a \wedge d$

$a \leftarrow g.$
$b \leftarrow j.$
$d \leftarrow m.$
$f \leftarrow m.$
$g \leftarrow m.$
$k \leftarrow m.$
$p.$