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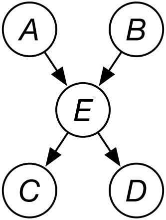
$\longrightarrow$ supervised learning

Learning belief network example

Model



Data

A	B	C	D	E
t	f	t	t	f
f	t	t	t	t
t	t	f	t	f
		$\dots$		

$\rightarrow$ Probabilities

$P(A)$
$P(B)$
$P(E \mid A, B)$
$P(C \mid E)$
$P(D \mid E)$

Learning conditional probabilities

- Each conditional probability distribution can be learned separately:
- For example:

$$P(E = t \mid A = t \wedge B = f) = \frac{(\#\text{examples: } E = t \wedge A = t \wedge B = f) + c_1}{(\#\text{examples: } A = t \wedge B = f) + c}$$

  where c_1 and c reflect prior (expert) knowledge ($c_1 \leq c$).
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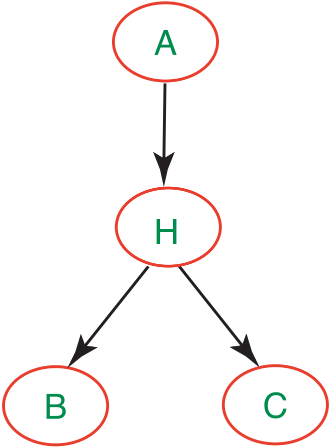
  where c_1 and c reflect prior (expert) knowledge ($c_1 \leq c$).
- When there are many parents to a node, there can little or no data for each conditional probability: use supervised learning to learn a decision tree, linear classifier, a neural network or other representation of the conditional probability.

Unobserved Variables

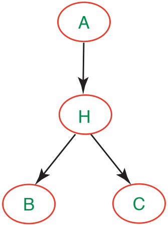


- What if you had only observed values for A, B, C?

A	B	C
t	f	t
f	t	t
t	t	f
	$\dots$	

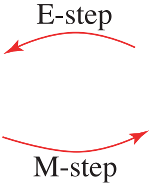
EM Algorithm

Model



Augmented Data

A	B	C	H	$Count$
t	f	t	t	0.7
t	f	t	f	0.3
f	t	t	f	0.9
f	t	t	t	0.1
	$\dots$			$\dots$



Probabilities

$P(A)$
$P(H \mid A)$
$P(B \mid H)$
$P(C \mid H)$

EM Algorithm

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- Start either with made-up data or made-up probabilities.
- EM will converge to a local maxima.

Belief network structure learning (I)

Given examples $\mathbf{e}$, and model m:

$$P(m \mid \mathbf{e}) = \frac{P(\mathbf{e} \mid m) * P(m)}{P(\mathbf{e})}.$$

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- Taking logarithms (and negating):

$$\arg\max_m P(m \mid \mathbf{e}) = \arg\min_m(-\log P(\mathbf{e} \mid m) - \log P(m))$$

Description Length

Bayes Rule:

$$P(h|d) \propto P(d|h)P(h)$$

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- $\log P(d|h)$ measures fit to data
- $\log P(h)$ measures model complexity

Information theory overview

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- Can you do better?

Information and Probability

Consider a code to distinguish elements of $\{a, b, c, d\}$ with

$$P(a) = \frac{1}{2}, P(b) = \frac{1}{4}, P(c) = \frac{1}{8}, P(d) = \frac{1}{8}$$

Consider the code:

a	0	b	10	c	110	d	111

This code uses bits.

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- The expected number of bits it takes to describe a distribution given evidence e:

$$I(e) = \sum_x -P(x|e) * \log_2 P(x|e).$$

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Comparing Distributions

- Suppose a code is designed to be optimal for probability distribution Q, so that x is described using $-log_2 Q(x)$ bits.
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- When is this large? When $P(x) \gg Q(x)$ for some x.

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  - How to determine $-\log P(m)$?

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- If there are $||m||$ independent parameters ($||m||$ is the dimensionality of the model):

$$-\log P(m \mid \mathbf{e}) \propto -\log P(\mathbf{e} \mid m) + ||m|| \log(|\mathbf{e}|)$$

This is the Bayesian Information Criterion (BIC) score.

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- In general you need to model why data is missing (see Chapter 11)

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... this is too difficult for current techniques!