Agents as Processes

Agents carry out actions:

- forever infinite horizon
- until some stopping criteria is met indefinite horizon
- finite and fixed number of steps finite horizon

Decision-theoretic Planning

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- actions can be stochastic; the outcome of an action can't be fully predicted
- there is a model that specifies the (probabilistic) outcome of actions and the rewards
- the world is fully observable?

Initial Assumptions

- flat or modular or hierarchical
- explicit states or features or individuals and relations
- static or finite stage or indefinite stage or infinite stage
- fully observable or partially observable
- deterministic or stochastic dynamics
- goals or complex preferences
- single agent or multiple agents
- knowledge is given or knowledge is learned
- perfect rationality or bounded rationality

Utility and time

- Would you prefer \$1000 today or \$1000 next year?
- What price would you pay now to have an eternity of happiness?
- How can you trade off pleasures today with pleasures in the future?

Utility and time

- How would you compare the following sequences of rewards (per week):

  A: \$1000000, \$0, \$0, \$0, \$0, \$0,...
  B: \$1000, \$1000, \$1000, \$1000, \$1000,...
  C: \$1000, \$0, \$0, \$0, \$0,...
  D: \$1, \$1, \$1, \$1, \$1,...
  E: \$1, \$2, \$3, \$4, \$5,...

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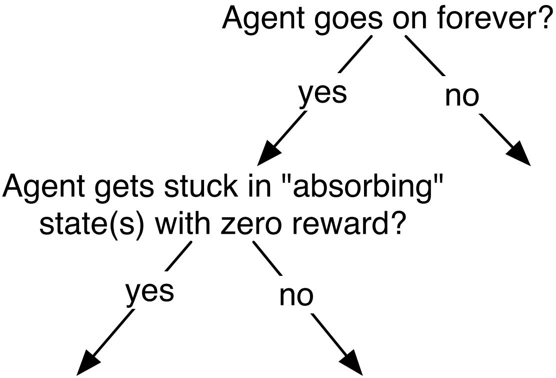
- total reward $V = \sum_{i=1}^{\infty} r_i$
- average reward $V = \lim_{n \to \infty} (r_1 + \cdots + r_n)/n$

Average vs Accumulated Rewards



Rewards and Values

Suppose the agent receives a sequence of rewards $r_1, r_2, r_3, r_4, \ldots$ in time.

- discounted return $V = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$
  γ is the discount factor $0 \leq \gamma \leq 1$.

Properties of the Discounted Rewards

- The discounted return for rewards $r_1, r_2, r_3, r_4, \ldots$ is

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- How is the infinite future valued compared to immediate rewards?
  $1 + \gamma + \gamma^2 + \gamma^3 + \cdots = 1/(1 - \gamma)$
  Therefore $\dfrac{\text{minimum reward}}{1-\gamma} \leq V_t \leq \dfrac{\text{maximum reward}}{1-\gamma}$
- We can approximate V with the first k terms, with error:

$$V - (r_1 + \gamma r_2 + \cdots + \gamma^{k-1} r_k) = \gamma^k V_{k+1}$$

World State

- The world state is the information such that if the agent knew the world state, no information about the past is relevant to the future. Markovian assumption.
- S_i is state at time i, and A_i is the action at time i:

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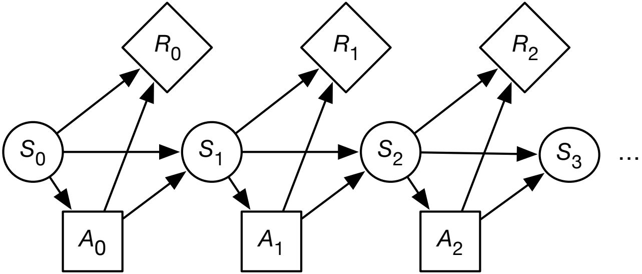
  $P(s' \mid s, a)$ is the probability that the agent will be in state s' immediately after doing action a in state s.
- The dynamics is stationary if the distribution is the same for each time point.

- A Markov decision process augments a Markov chain with actions and rewards:



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- γ is discount factor.

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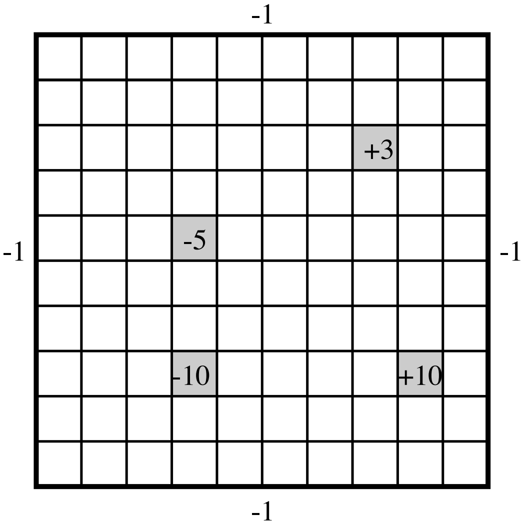
State	Action	Reward
healthy	relax	7
healthy	party	10
sick	relax	0
sick	party	2

Example: Simple Grid World



Grid World Model

- Actions: up, down, left, right.
- 100 states corresponding to the positions of the robot.
- Robot goes in the commanded direction with probability 0.7, and one of the other directions with probability 0.1.
- If it crashes into an outside wall, it remains in its current position and has a reward of -1.
- Four special rewarding states; the agent gets the reward when leaving.

Planning Horizons

The planning horizon is how far ahead the planner looks to make a decision.

- The robot gets flung to one of the corners at random after leaving a positive ($+10$ or $+3$) reward state.
  - the process never halts
  - infinite horizon

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  - the process never halts
  - infinite horizon
- The robot gets +10 or +3 in the state, then it stays there getting no reward. These are absorbing states.
  - The robot will eventually reach an absorbing state.
  - indefinite horizon

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What information is available when the agent decides what to do?

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[This lecture only considers FOMDPs.
POMDPS are much harder to solve.]

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  (A randomized policy or a non-stationary policy is never better than a stationary policy.)

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For the grid world with 100 states and 4 actions,
how many stationary policies are there?

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$$\begin{aligned} V^\pi(s) &= Q^\pi(s,\pi(s)) \\ Q^\pi(s,a) &= R(s,a) + \gamma \sum_{s'} P(s' \mid a,s) V^\pi(s') \end{aligned}$$

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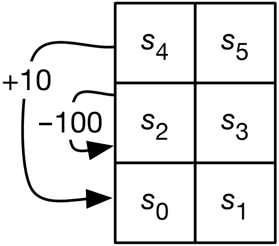
  The error reduces proportionally to $\dfrac{\gamma^k}{1-\gamma}$

Tiny MDP Example: 6 states, 4 actions



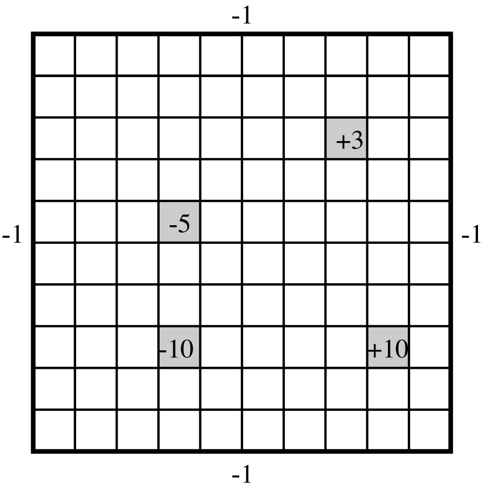
Actions (if crash, stay still with a reward of -1)

- ➡ move right
- ⬅ move left, except as above.
- ⬆ up risky: $P(up)$=0.8, $P(left)$=0.1, $P(right)$=0.1
- ⬇ up carefully: go up with extra reward of -1

AIPython: `python -i mdpExamples.py`

```
MDPtiny(discount=0.9).show()
```

Example: Simple Grid World



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grid(discount=0.9).show()
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- This converges to the optimal value functions, if each state and action is visited infinitely often in the limit.
- It can either store $V[s]$ or $Q[s, a]$.

Asynchronous VI: storing $V[s]$

- Repeat forever:
  - Select state s
  - $V[s] \leftarrow$

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Policy Iteration

- Set π_0 arbitrarily, let $i = 0$
- Repeat:
  - evaluate $Q^{\pi_i}(s, a)$
  - let $\pi_{i+1}(s) = argmax_a Q^{\pi_i}(s, a)$
  - set $i = i + 1$
- until $\pi_i(s) = \pi_{i-1}(s)$

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Evaluating $Q^{\pi_i}(s, a)$ means finding a solution to a set of $|S| \times |A|$ linear equations with $|S| \times |A|$ unknowns.

It can also be approximated iteratively.

Modified Policy Iteration

Set $\pi[s]$ arbitrarily
Set $Q[s,a]$ arbitrarily
Repeat forever:

- Repeat for a while:
  - Select state s, action a
  - $Q[s,a] \leftarrow R(s,a) + \gamma \sum_{s'} P(s' \mid s,a) Q[s', \pi[s']]$
- $\pi[s] \leftarrow argmax_a Q[s,a]$

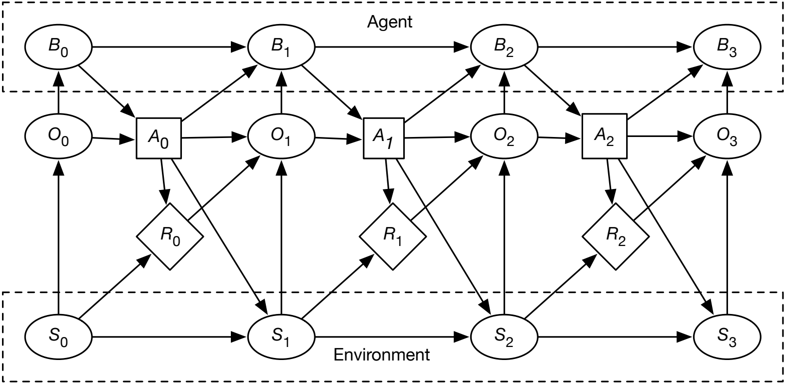
Q, V, π, R

$$Q^*(s,a) = \sum_{s'} P(s' \mid a, s) \left(R(s,a,s') + \gamma V^*(s')\right)$$
$$= R(s,a) + \gamma \sum_{s'} P(s' \mid a, s) V^*(s')$$
$$V^*(s) = \max_a Q^*(s,a)$$
$$\pi^*(s) = argmax_a Q^*(s,a)$$

where

$$R(s,a) = \sum_{s'} P(s' \mid a, s) R(s,a,s')$$

Partially Observable MDP (POMDP)



B_i agent's belief state at time i. A_i agent's action. O_i is what the agent observes. R_i is the reward. S_i is the world state.