# Learning Objectives

At the end of the class you should be able to:

- recognize and represent constraint satisfaction problems
- show how constraint satisfaction problems can be solved with search
- implement and trace arc-consistency of a constraint graph
- show how domain splitting can solve constraint problems

# Posing a Constraint Satisfaction Problem

A CSP is characterized by

- A set of variables $V_1, V_2, \ldots, V_n$.
- Each variable $V_i$ has an associated domain $\mathbf{D}_{V_i}$ of possible values.
- There are hard constraints on various subsets of the variables which specify legal combinations of values for these variables.
- A solution to the CSP is an assignment of a value to each variable that satisfies all the constraints.

# Example: scheduling activities

- Variables: $A$, $B$, $C$, $D$, $E$ that represent the starting times of various activities.
- Domains: $\mathbf{D}_A = \{1,2,3,4\}$, $\mathbf{D}_B = \{1,2,3,4\}$, $\mathbf{D}_C = \{1,2,3,4\}$, $\mathbf{D}_D = \{1,2,3,4\}$, $\mathbf{D}_E = \{1,2,3,4\}$
- Constraints:

$$
\begin{aligned}
&(B \neq 3) \wedge (C \neq 2) \wedge (A \neq B) \wedge (B \neq C) \wedge \\
&\quad (C < D) \wedge (A = D) \wedge (E < A) \wedge (E < B) \wedge \\
&\quad (E < C) \wedge (E < D) \wedge (B \neq D).
\end{aligned}
$$

# Generate-and-Test Algorithm

- Generate the assignment space $\mathbf{D} = \mathbf{D}_{V_1} \times \mathbf{D}_{V_2} \times \ldots \times \mathbf{D}_{V_n}$. Test each assignment with the constraints.
- Example:

$$\begin{aligned}
\mathbf{D} &= \mathbf{D}_A \times \mathbf{D}_B \times \mathbf{D}_C \times \mathbf{D}_D \times \mathbf{D}_E \\
&= \{1,2,3,4\} \times \{1,2,3,4\} \times \{1,2,3,4\} \\
&\quad \times \{1,2,3,4\} \times \{1,2,3,4\} \\
&= \{\langle 1,1,1,1,1\rangle, \langle 1,1,1,1,2\rangle, ..., \langle 4,4,4,4,4\rangle\}.
\end{aligned}$$

- How many assignments need to be tested for $n$ variables each with domain size $d$?

# Backtracking Algorithms

- Systematically explore **D** by instantiating the variables one at a time
- evaluate each constraint predicate as soon as all its variables are bound
- any partial assignment that doesn't satisfy the constraint can be pruned.

Example Assignment $A = 1 \wedge B = 1$ is inconsistent with constraint $A \neq B$ regardless of the value of the other variables.

# CSP as Graph Searching

A CSP can be solved by graph-searching:

- A node is an assignment values to some of the variables.
- Suppose node $N$ is the assignment $X_1 = v_1, \ldots, X_k = v_k$.
  Select a variable $Y$ that isn't assigned in $N$.
  For each value $y_i \in dom(Y)$
  $X_1 = v_1, \ldots, X_k = v_k, Y = y_i$ is a neighbour if it is consistent with the constraints.
- The start node is the empty assignment.
- A goal node is a total assignment that satisfies the constraints.

# Consistency Algorithms

- Idea: prune the domains as much as possible before selecting values from them.
- A variable is **domain consistent** if no value of the domain of the node is ruled impossible by any of the constraints.
- **Example:** Is the scheduling example domain consistent?

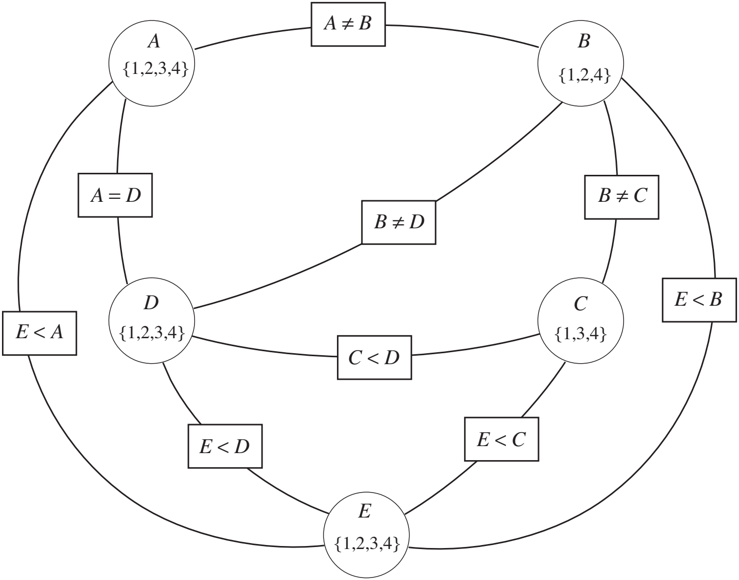
# Consistency Algorithms

- Idea: prune the domains as much as possible before selecting values from them.
- A variable is domain consistent if no value of the domain of the node is ruled impossible by any of the constraints.
- Example: Is the scheduling example domain consistent? $\mathbf{D}_B = \{1, 2, 3, 4\}$ isn't domain consistent as $B = 3$ violates the constraint $B \neq 3$.

# Constraint Network

- There is a oval-shaped node for each variable.
- There is a rectangular node for each constraint.
- There is a domain of values associated with each variable node.
- There is an arc from variable $X$ to each constraint that involves $X$.

# Example Constraint Network



# Arc Consistency

- An arc $\langle X, r(X, \overline{Y}) \rangle$ is arc consistent if, for each value $x \in dom(X)$, there is some value $\overline{y} \in dom(\overline{Y})$ such that $r(x, \overline{y})$ is satisfied.
- A network is arc consistent if all its arcs are arc consistent.
- What if arc $\langle X, r(X, \overline{Y}) \rangle$ is *not* arc consistent?

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- A network is arc consistent if all its arcs are arc consistent.
- What if arc $\langle X, r(X, \overline{Y}) \rangle$ is *not* arc consistent?
  All values of $X$ in $dom(X)$ for which there is no corresponding value in $dom(\overline{Y})$ can be deleted from $dom(X)$ to make the arc $\langle X, r(X, \overline{Y}) \rangle$ consistent.

# Arc Consistency Algorithm

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- When an arc has been made arc consistent, does it ever need to be checked again?

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  An arc $\langle X, r(X, \overline{Y}) \rangle$ needs to be revisited if the domain of one of the $Y$'s is reduced.
- Three possible outcomes when all arcs are made arc consistent: (Is there a solution?)
  - One domain is empty $\Longrightarrow$
  - Each domain has a single value $\Longrightarrow$
  - Some domains have more than one value $\Longrightarrow$

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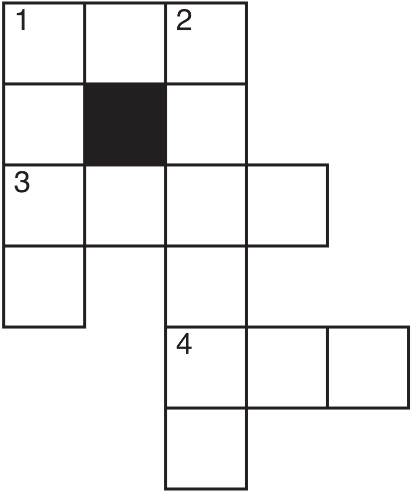
  An arc $\langle X, r(X, \overline{Y}) \rangle$ needs to be revisited if the domain of one of the $Y$'s is reduced.
- Three possible outcomes when all arcs are made arc consistent: (Is there a solution?)
  - One domain is empty $\Longrightarrow$ no solution
  - Each domain has a single value $\Longrightarrow$ unique solution
  - Some domains have more than one value $\Longrightarrow$ there may or may not be a solution

# Finding solutions when AC finishes

- If some domains have more than one element $\Longrightarrow$ search
- Split a domain, then recursively solve each half.
- It is often best to split a domain in half.
- Do we need to restart from scratch?

# Example: Crossword Puzzle




**Words:**

ant, big, bus, car, has
book, buys, hold,
lane, year
beast, ginger, search,
symbol, syntax

# Hard and Soft Constraints

- Given a set of variables, assign a value to each variable that either
  - satisfies some set of constraints: satisfiability problems — "hard constraints"
  - minimizes some cost function, where each assignment of values to variables has some cost: optimization problems — "soft constraints"
- Many problems are a mix of hard and soft constraints (called constrained optimization problems).