Agents as Processes

Agents carry out actions:

- forever **infinite horizon**
- until some stopping criteria is met **indefinite horizon**
- finite and fixed number of steps **finite horizon**

Decision-theoretic Planning

What should an agent do when

- it gets rewards (and punishments) and tries to maximize its rewards received
- actions can be stochastic; the outcome of an action can't be fully predicted
- there is a model that specifies the (probabilistic) outcome of actions and the rewards
- the world is fully observable

Initial Assumptions

- flat or modular or hierarchical
- explicit states or features or individuals and relations
- static or finite stage or indefinite stage or infinite stage
- fully observable or partially observable
- deterministic or stochastic dynamics
- goals or complex preferences
- single agent or multiple agents
- knowledge is given or knowledge is learned
- perfect rationality or bounded rationality

Utility and time

- Would you prefer \$1000 today or \$1000 next year?
- What price would you pay now to have an eternity of happiness?
- How can you trade off pleasures today with pleasures in the future?

Utility and time

- How would you compare the following sequences of rewards (per week):

  A: \$1000000, \$0, \$0, \$0, \$0, \$0,...
  B: \$1000, \$1000, \$1000, \$1000, \$1000,...
  C: \$1000, \$0, \$0, \$0, \$0,...
  D: \$1, \$1, \$1, \$1, \$1,...
  E: \$1, \$2, \$3, \$4, \$5,...

Rewards and Values

Suppose the agent receives a sequence of rewards $r_1, r_2, r_3, r_4, \dots$ in time. What utility should be assigned?
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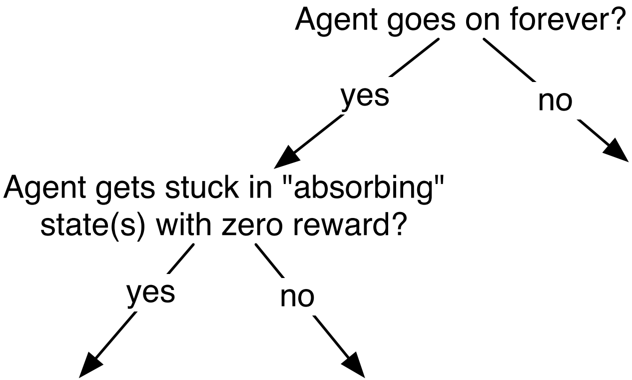
- total reward $V = \sum_{i=1}^{\infty} r_i$
- average reward $V = \lim_{n \to \infty} (r_1 + \cdots + r_n)/n$

Average vs Accumulated Rewards



Rewards and Values

Suppose the agent receives a sequence of rewards $r_1, r_2, r_3, r_4, \ldots$ in time.

- discounted return $V = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$
  γ is the discount factor $0 \le \gamma \le 1$.

Properties of the Discounted Rewards

- The discounted return for rewards $r_1, r_2, r_3, r_4, \ldots$ is

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- If V_t is the value obtained from time step t
$$V_t = r_t + \gamma V_{t+1}$$
- How is the infinite future valued compared to immediate rewards?
$1 + \gamma + \gamma^2 + \gamma^3 + \cdots = 1/(1-\gamma)$
Therefore $\dfrac{\text{minimum reward}}{1-\gamma} \leq V_t \leq \dfrac{\text{maximum reward}}{1-\gamma}$
- We can approximate V with the first k terms, with error:
$$V - (r_1 + \gamma r_2 + \cdots + \gamma^{k-1} r_k) = \gamma^k V_{k+1}$$

World State

- The world state is the information such that if the agent knew the world state, no information about the past is relevant to the future. Markovian assumption.
- S_i is state at time i, and A_i is the action at time i:

$$P(S_{t+1}|S_0, A_0, \ldots, S_t, A_t) =$$

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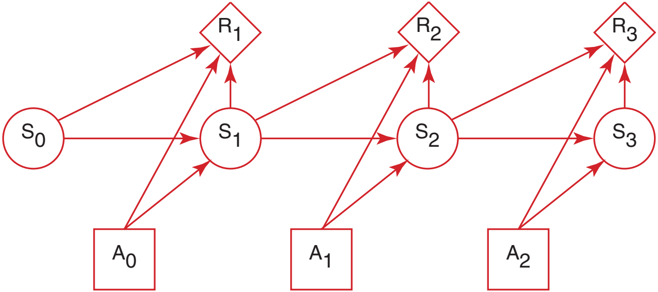
$$P(S_{t+1}|S_0, A_0, \ldots, S_t, A_t) = P(S_{t+1}|S_t, A_t)$$

  $P(s'|s, a)$ is the probability that the agent will be in state s' immediately after doing action a in state s.
- The dynamics is stationary if the distribution is the same for each time point.

Decision Processes

- A Markov decision process augments a Markov chain with actions and values:



Markov Decision Processes

An MDP consists of:

- set S of states.
- set A of actions.
- $P(S_{t+1}|S_t, A_t)$ specifies the dynamics.
- $R(S_t, A_t, S_{t+1})$ specifies the reward at time t.
  $R(s, a, s')$ is the expected reward received when the agent is in state s, does action a and ends up in state s'.
- γ is discount factor.

Example: to exercise or not?

Each week *Sam* has to decide whether to exercise or not:

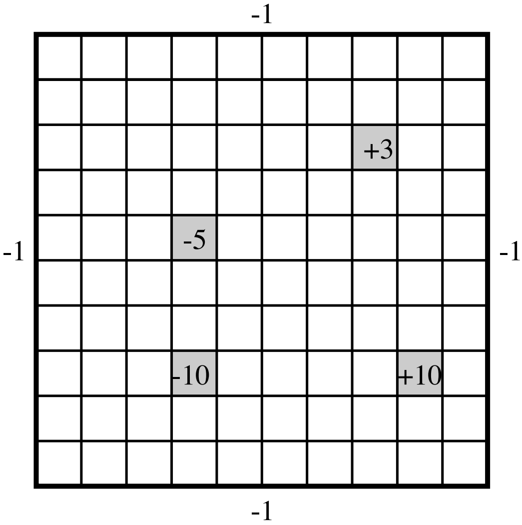
- States: $\{fit, unfit\}$
- Actions: $\{exercise, relax\}$
- Dynamics:

State	Action	$P(fit \mid State, Action)$
fit	exercise	0.99
fit	relax	0.7
unfit	exercise	0.2
unfit	relax	0.0

- Reward (does not depend on resulting state):

State	Action	Reward
fit	exercise	8
fit	relax	10
unfit	exercise	0
unfit	relax	5

Example: Simple Grid World



Grid World Model

- Actions: up, down, left, right.
- 100 states corresponding to the positions of the robot.
- Robot goes in the commanded direction with probability 0.7, and one of the other directions with probability 0.1.
- If it crashes into an outside wall, it remains in its current position and has a reward of -1.
- Four special rewarding states; the agent gets the reward when leaving.

Planning Horizons

The planning horizon is how far ahead the planner looks to make a decision.

- The robot gets flung to one of the corners at random after leaving a positive (+10 or +3) reward state.
  - the process never halts
  - infinite horizon
- The robot gets +10 or +3 in the state, then it stays there getting no reward. These are absorbing states.
  - The robot will eventually reach an absorbing state.
  - indefinite horizon

Information Availability

What information is available when the agent decides what to do?

- fully-observable MDP the agent gets to observe S_t when deciding on action A_t.
- partially-observable MDP (POMDP) the agent has some noisy sensor of the state. It needs to remember its sensing and acting history.

[This lecture only considers FOMDPs]

Policies

- A stationary policy is a function:

$$\pi : S \rightarrow A$$

  Given a state s, $\pi(s)$ specifies what action the agent who is following π will do.
- An optimal policy is one with maximum expected discounted reward.
- For a fully-observable MDP with stationary dynamics and rewards with infinite or indefinite horizon, there is always an optimal stationary policy.

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How many stationary policies are there?
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For the grid world with 100 states and 4 actions,
how many stationary policies are there?

Value of a Policy

Given a policy π:

- $Q^\pi(s, a)$, where a is an action and s is a state, is the expected value of doing a in state s, then following policy π.
- $V^\pi(s)$, where s is a state, is the expected value of following policy π in state s.
- Q^π and V^π can be defined mutually recursively:

$$\begin{aligned} Q^\pi(s, a) &= \\ V^\pi(s) &= \end{aligned}$$

Value of the Optimal Policy

- $Q^*(s,a)$, where a is an action and s is a state, is the expected value of doing a in state s, then following the optimal policy.
- $V^*(s)$, where s is a state, is the expected value of following the optimal policy in state s.
- Q^* and V^* can be defined mutually recursively:

$$
\begin{aligned}
Q^*(s,a) &= \\
V^*(s) &= \\
\pi^*(s) &=
\end{aligned}
$$

Value Iteration

- Let V_k and Q_k be k-step lookahead value and Q functions.
- Idea: Given an estimate of the k-step lookahead value function, determine the $k+1$ step lookahead value function.
- Set V_0 arbitrarily.
- Compute Q_{i+1}, V_{i+1} from V_i.
- This converges exponentially fast (in k) to the optimal value function.

  The error reduces proportionally to $\dfrac{\gamma^k}{1-\gamma}$

Asynchronous Value Iteration

- The agent doesn't need to sweep through all the states, but can update the value functions for each state individually.
- This converges to the optimal value functions, if each state and action is visited infinitely often in the limit.
- It can either store $V[s]$ or $Q[s, a]$.

Asynchronous VI: storing $V[s]$

- Repeat forever:
  - Select state s
  - $V[s] \leftarrow \max_a \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma V[s']\right)$

Asynchronous VI: storing $Q[s, a]$

- Repeat forever:
  - Select state s, action a
  - $Q[s, a] \leftarrow \sum_{s'} P(s'|s, a) \left(R(s, a, s') + \gamma \max_{a'} Q[s', a'] \right)$

Policy Iteration

- Set π_0 arbitrarily, let $i = 0$
- Repeat:
  - evaluate $Q^{\pi_i}(s, a)$
  - let $\pi_{i+1}(s) = argmax_a Q^{\pi_i}(s, a)$
  - set $i = i + 1$
- until $\pi_i(s) = \pi_{i-1}(s)$

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Evaluating $Q^{\pi_i}(s, a)$ means finding a solution to a set of $|S| \times |A|$ linear equations with $|S| \times |A|$ unknowns.

It can also be approximated iteratively.

Modified Policy Iteration

Set $\pi[s]$ arbitrarily
Set $Q[s,a]$ arbitrarily
Repeat forever:

- Repeat for a while:
  - Select state s, action a
  - $Q[s,a] \leftarrow \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma Q[s',\pi[s']]\right)$
- $\pi[s] \leftarrow argmax_a Q[s,a]$

Q, V, π, R

$$
\begin{aligned}
Q^*(s,a) &= \sum_{s'} P(s'|a,s)\left(R(s,a,s') + \gamma V^*(s')\right) \\
V^*(s) &= \max_a Q(s,a) \\
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\end{aligned}
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Let

$$
R(s,a) = \sum_{s'} P(s'|a,s) R(s,a,s')
$$

Then:

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